Tight Bounds on the Laplace Transform of Interference in a Poisson Hole Process

Zeinab Yazdanshenasan, Harpreet S. Dhillon, Mehrnaz Afshang, and Peter Han Joo Chong

***Abstract*—To maintain tractability, interference field is often modeled as a homogeneous Poisson Point Process (PPP) in the analysis of wireless networks. While it provides meaningful first-order results, it falls short in modeling the effect of interference management techniques, which typically introduce some form of *spatial interaction* among transmitters. In some applications, such as cognitive radio and device-to-device networks, this interaction results in the formation of holes in an otherwise homogeneous interference field. The resulting interference field can be accurately modeled as a *Poisson Hole Process (PHP)*. Despite the importance of PHP in modeling wireless networks, exact characterization of the interference experienced by a typical node in a PHP is an open problem. In this paper, we introduce a new approach to modeling the PHP, in which we *dissolve* the holes in such a way that it results in an equivalent non-homogeneous PPP, which is much more amenable to shot-noise analysis. Using this approach, we derive new lower and upper bounds on the Laplace transform of interference in a PHP. The new bounds are compared numerically to the known approaches and are shown to be very tight under various operational regimes.**

***Index Terms*—Interference modeling, stochastic geometry, Poisson Hole Process, coverage probability.**

I. Introduction

Stochastic geometry has emerged as an important tool for the analysis of wireless networks [1]. Initially popular for the modeling of ad hoc and wireless sensor networks, it has recently been adopted for the analysis of cellular and heterogeneous cellular networks as well [2], [3]. Irrespective of the nature of the wireless network, the interference field is almost always modeled by a homogeneous PPP to maintain tractability. While this leads to remarkably simple results for key performance metrics, such as coverage and rate, it is not quite suited to model the effect of interference management techniques, which often introduce some form of spatial interaction among transmitters. In this paper, we focus on *spatial separation*, where holes (also called *exclusion zones*) are created around nodes/networks that need to be protected from excessive interference [4]. In particular, we assume that the *baseline* interference field is a PPP from which holes of a given radius are carved out. When the locations of the holes also form an independent PPP, the resulting point process is usually termed as a PHP, which is the main focus of this paper.

There are numerous instances in wireless networks where PHP is the more appropriate model for node locations. In cognitive radio networks, holes are established around primary links, which means the secondary transmitters that are not inside the holes form a PHP. In [5], the performance of a cognitive ad hoc network is studied under this model. In addition, PHP also appears in a two tier Heterogeneous network comprising macro-cells and small cells which operate under the spectrum underlay strategy [6]–[8]. Here, interference mitigation schemes in a two-tier heterogeneous network establish holes around the active nodes of the macro-tier whereas the remaining small cell transmitters form a PHP. In addition, for underlay D2D communication in cellular networks, inhibition zones may be created around cellular links where no D2D transmissions are allowed, thus saving cellular links from excessive D2D interference. The active D2D transmitters outside the holes form PHP [9], [10]. In this regard, cognitive D2D communication in cellular network when transmitters are powered by harvesting energy from the ambient interference is studied in [11]. In [12], a Poisson Cluster Process (PCP) and a PHP are merged to develop a new spatial model for integrated D2D and cellular networks. In particular, a modified Thomas cluster process is used to model device locations where instead of modeling the cluster centers as a homogeneous PPP, they are modeled as a PHP to account for the inhibition zones around cellular links.

Although PHP model has wide applicability in wireless networks, its probability generating functional (PGFL) is not easy to characterize, which makes its performance analysis more challenging. There are two main directions taken in the literature for the analysis of wireless networks modeled by PHPs. The first approach, termed *first-order statistic approximation*, approximates PHP by a homogeneous PPP with the same density [13]. The second approach ignores the holes altogether to approximate the PHP by its *baseline* PPP. This overestimates the interference and the accuracy of the approximation is a function of system parameters [5], [9]–[11]. Besides, the PHP is sometimes approximated with a PCP by matching the first and second order statistics [5]–[7]. The resulting expressions for performance metrics are usually more complicated in this case compared to the above two.

Contributions and outcomes. The key contributions of this paper are the tight provable bounds on the Laplace transform of interference at a typical node of a PHP. We first consider only one hole in the interference field; the one that is closest to the typical node; and derive a lower bound on the Laplace transform. We then argue that this hole can be *dissolved* in such way that the original interference field with a hole can be equivalently modeled by a non-homogeneous PPP. Using this insight, we derive a provable upper bound on the Laplace

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transform by considering all the holes of a PHP. The tightness of these bounds is analytically demonstrated by showing that their ratio is very close to one over a wide range of parameters. The proposed bounds are also compared numerically with the existing approaches. Since our bounds are derived without distorting the structure of holes through approximations, they work significantly better than any known approach. Further, it is well known that under Rayleigh fading, coverage probability reduces to being the Laplace transform of interference [13], which means these results also provide provable bounds on the coverage probability of a typical node in a PHP.

II. Network Model

A. System Model

We consider a wireless network that is modeled by a PHP in $\mathbb{R}^2$. A PHP can be formally defined in terms of two independent homogeneous PPPs Φ_1 and Φ_2, where Φ_2 represents the *baseline* PPP from which the holes are carved out and Φ_1 represents the locations of the holes. Let the densities of Φ_1 and Φ_2 be λ_1 and λ_2, respectively, with $\lambda_2 > \lambda_1$. Denoting the radius of each hole by D, the region covered by the holes can be expressed as

$$\Xi_D \triangleq \bigcup_{\mathbf{y}\in\Phi_1} \mathbf{b}(\mathbf{y}, D), \quad \mathbf{b}(\mathbf{y}, D) \equiv \{\mathbf{x} : \|\mathbf{x} - \mathbf{y}\| < D\}. \quad (1)$$

The points of Φ_2 lying outside Ξ_D, form a PHP, which can be formally expressed as

$$\Psi = \{\mathbf{x} \in \Phi_2 : \mathbf{x} \notin \Xi_D\} = \Phi_2 \setminus \Xi_D. \quad (2)$$

Note that the PHP has also been known as a Hole-1 process in the literature [14]. We characterize the interference experienced by a typical node in Ψ due to the transmission of the other nodes of Ψ. Due to the stationarity of the process, the typical node can be assumed to lie at the origin $\mathbf{o}$, and due to Slivnyak's theorem, we can condition on $\mathbf{o} \in \Psi$ without changing the distribution of the rest of the process [13]. Since the typical point is outside the holes by construction, there are no points of Φ_1 within a disk of radius D around the typical point. We assume that the serving transmitter is located at a fixed distance r_0. All the transmitters transmit at the same power P. For the wireless channel, we consider a standard power law path-loss with exponent $\alpha > 2$ along with independent Rayleigh fading. Hence, the received power at a typical receiver from its serving node is $P_r = Phr_0^{-\alpha}$, where $h \sim \exp(1)$ models Rayleigh fading. Similarly, the interference power experienced by the typical receiver is

$$I = \sum_{\mathbf{x}\in\Psi} Ph_{\mathbf{x}} \|\mathbf{x}\|^{-\alpha}, \quad (3)$$

where $h_{\mathbf{x}} \sim \exp(1)$ models Rayleigh fading gain for the link from interferer $\mathbf{x} \in \Psi$ to the typical receiver.

B. SIR and Coverage Probability

Using the received power over the link of interest and the interference power defined in the previous subsection, the *signal to interference ratio* (SIR) can be expressed as:

$$\mathtt{SIR}(r_0) = \frac{Phr_0{}^{-\alpha}}{\sum_{\mathbf{x}\in\Psi} Ph_{\mathbf{x}} \|\mathbf{x}\|^{-\alpha}}. \quad (4)$$

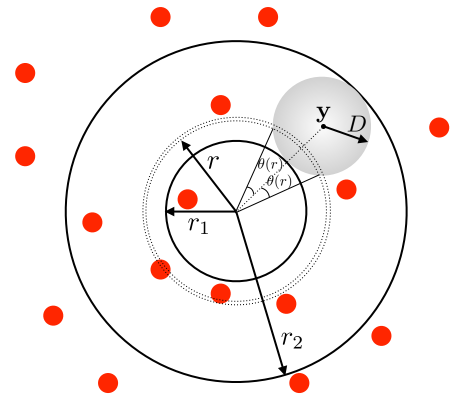

Fig. 1. Geometrical model of interference field involving a single hole.

Denote the minimum SIR required for successful decoding and demodulation at the typical receiver by γ. A useful metric of interest in wireless networks is the SIR *coverage probability* $\mathtt{P_c}$, which is the probability that the SIR at the receiver exceeds the threshold γ. Mathematically, it is

$$\mathtt{P_c} = \mathbb{P}\{\mathtt{SIR}(r_0) > \gamma\} = \mathbb{P}\left\{h > \frac{\gamma r_0^\alpha}{P} I\right\}$$
$$\stackrel{(a)}{=} \mathbb{E}\left[\exp\left(-\frac{\gamma r_0^\alpha}{P} I\right)\right] \stackrel{(b)}{=} \mathcal{L}_I\left(\frac{\gamma r_0^\alpha}{P}\right), \quad (5)$$

where (a) follows from the fact that $h \sim \exp(1)$, and (b) from the definition of Laplace transform of interference power $\mathcal{L}_I(s) = \mathbb{E}[\exp(-sI)]$. Clearly, it is sufficient to focus on the Laplace transform of interference to study coverage probability which can be easily derived for our setup using (5). The Laplace transform $\mathcal{L}_I(s)$ is characterized in the next Section.

III. Laplace Transform of Interference in a PHP

Two popular approaches have been used in the literature to derive the Laplace transform of interference in a PHP. The first approach is to ignore the holes and approximate the interference field Ψ by the baseline PPP Φ_2 of density λ_2. This approach clearly overestimates the interference power and hence leads to the lower bound on the Laplace transform of interference [5]. In the second approach, termed *first-order statistic approximation*, the baseline PPP is independently thinned such that the resulting density of the PPP is the same as that of a PHP [13]. Note that the density of a PHP in terms of λ_1, λ_2, and D can be shown to be $\lambda_{\mathrm{PHP}} = \lambda_2 \exp(-\lambda_1 \pi D^2)$ [13]. Since independent and uniform thinning of a PPP may remove dominant interferers in certain instances, this approach is expected to underestimate the interference and consequently overestimate the coverage probability. We will compare these approaches with our proposed approach in the numerical results section. We now present our proposed approach to characterizing the Laplace transform of interference in a PHP. In the first step, we model the locations of interferers by a homogeneous PPP Φ_2 of density λ_2 from which one single deterministic hole $\mathcal{C}$ of radius D is removed. Let the location of the center of this hole be $\mathbf{y} \in \mathbb{R}$ and hence its distance from the origin be $\|\mathbf{y}\|$. The resulting setup is illustrated in Fig. 1. Note that the interference field in this case is non-isotropic due to the fixed location of the hole. The Laplace transform of the interference power received at the origin from the nodes of Φ_2 outside $\mathcal{C}$ is characterized in the next Lemma.

Lemma 1. *Let $I = \sum_{\mathbf{x}\in\Phi_2\cap\mathbf{b}^c(\mathbf{y},D)} Ph_{\mathbf{x}}\|\mathbf{x}\|^{-\alpha}$, the Laplace transform of interference conditioned on $\|\mathbf{y}\|$ is $\mathcal{L}_{I|\|\mathbf{y}\|}(s) =$*

$$\exp\left(-\pi\lambda_2\frac{(sP)^{2/\alpha}}{\mathrm{sinc}(2/\alpha)}\right)\exp\left(\int_{\|\mathbf{y}\|-D}^{\|\mathbf{y}\|+D}\frac{2\pi\lambda(r)}{1+\frac{r^\alpha}{sP}}r\mathrm{d}r\right) \quad (6)$$

where $\lambda(r) = \frac{\lambda_2}{\pi}\arccos\left(\frac{r^2+\|\mathbf{y}\|^2-D^2}{2\|\mathbf{y}\|r}\right)$, and $\mathcal{C} = \mathbf{b}(\mathbf{y},D)$ denotes the hole centered at $\mathbf{y}$ with radius D.

Proof. See Appendix A. ■

Remark 1 (Dissolving the hole)**.** *The above result has an interesting interpretation that will be useful in visualizing the proposed results. Note that since received power is a radially symmetric function, it solely depends upon the distance of the transmitter to the origin. Therefore, we can in principle, dissolve the hole as long as the number of points lying in a thin strip of radius $\|\mathbf{y}\|-D \le r \le \|\mathbf{y}\|+D$ and vanishingly small width $\mathrm{d}r$ is not changed. Please refer to Fig. 1 for an illustration of this strip. Taking a closer look at the interference originating from this strip we note that the only thing that matters is the number of points that lie in the part of the strip that is outside the hole. The area of this region is $2r\mathrm{d}r(\pi-\theta(r))$, where the angle $\theta(r) = \arccos\left(\frac{r^2+\|\mathbf{y}\|^2-D^2}{2\|\mathbf{y}\|r}\right)$ is defined in Fig. 1. Therefore, the number of interfering points lying within this strip is Poisson distributed with mean $\lambda_2 2r\mathrm{d}r(\pi-\theta(r))$. Since the exact locations of these points within the strip doesn't matter, we can dissolve the hole and redistribute the points uniformly inside the whole strip of area $2\pi r\mathrm{d}r$. This means, the PPP with a hole can be equivalently modeled as a non-homogeneous PPP with density $\lambda_2(1-\theta(r)/\pi)$, where the $\lambda_2\theta(r)/\pi$ term (defined as $\lambda(r)$ in Lemma 1) captures the effect of hole.*

Using these insights, we now derive tight bounds on the Laplace transform of interference originating from the PHP.

A. Lower Bound on the Laplace Transform of Interference

Before going into the technical details, note that due to path-loss, the effect of holes that are close to the typical point will be much more significant compared to the holes that are farther away. Therefore, to derive an easy-to-use lower bound on the Laplace transform of interference, we consider only one hole; the one that is closest to the typical point; and ignore the other holes. Denoting the location of the closest hole by $\mathbf{y}_1$, the interference field in this case is $\Phi_2\cap\mathbf{b}^c(\mathbf{y}_1,D)\supset\Psi$, which clearly overestimates the interference of PHP and hence leads to a lower bound on the Laplace transform. Note that in Lemma 1, we have already derived the *conditional* Laplace transform for the case when there is one hole and its distance to the origin is known. To derive a lower bound, we simply need to assume this hole to be the closest point of Φ_1 to the origin and decondition the result of Lemma 1 with respect to the distribution of $V_1 = \|\mathbf{y}_1\|$. Since the typical point always lies outside the holes (see Section II), the closest point of Φ_1 is at least a distance D from it. Using this fact, the probability density function (PDF) of V_1 can be shown to be

$$f_{V_1}(v_1) = 2\pi\lambda_1 v_1\exp(-\pi\lambda_1(v_1^2-D^2)), \quad v_1\ge D. \quad (7)$$

Deconditioning the result of Lemma 1 with respect to this distribution, the proposed lower bound is derived below.

Theorem 1 (New lower bound)**.** *Let $I=\sum_{\mathbf{x}\in\Psi}Ph_{\mathbf{x}}\|\mathbf{x}\|^{-\alpha}$, the Laplace transform of interference is lower bounded by*

$$\mathcal{L}_I(s)\ge\exp\left(-\pi\lambda_2\frac{(sP)^{2/\alpha}}{\mathrm{sinc}(2/\alpha)}\right)\times \quad (8)$$
$$\int_D^\infty\exp\left(g(v_1)\right)2\pi\lambda_1 v_1\exp(-\pi\lambda_1(v_1^2-D^2))\mathrm{d}v_1$$

where $g(v_1)=\int_{v_1-D}^{v_1+D}\arccos\left(\frac{r^2+v_1^2-D^2}{2v_1 r}\right)\frac{2\lambda_2}{1+\frac{r^\alpha}{sP}}r\mathrm{d}r$.

Proof. See Appendix B. ■

The tightness of this bound will be demonstrated later in this section after deriving the upper bound.

B. Upper Bound for the Laplace Transform of Interference

To derive an upper bound on the Laplace transform, we handle each hole individually using the above approach. Note that since the centers of the holes follow a PPP Φ_1, there will obviously be overlaps among holes. Therefore, when we remove points of Φ_2 corresponding to each hole individually (without accounting for the overlaps), we may remove certain points multiple times thus underestimating the interference field, which results in an upper bound on the Laplace transform of interference. In the extended version of this paper, we show that handling the overlaps accurately leads to a significant loss in tractability but fortunately, the following upper bound derived by ignoring overlaps is fairly accurate.

Theorem 2 (New upper bound)**.** *The Laplace transform of interference in a PHP is upper bounded by*

$$\mathcal{L}_I(s)\le\exp\left(-\pi\lambda_2\frac{(sP)^{2/\alpha}}{\mathrm{sinc}(2/\alpha)}\right)\times$$
$$\exp\left(-2\pi\lambda_1\left(\int_D^\infty\left(1-\exp\left(f(v)\right)\right)v\mathrm{d}v\right)\right) \quad (9)$$

where $f(v)=2\lambda_2\int_{v-D}^{v+D}\arccos\left(\frac{r^2+v^2-D^2}{2vr}\right)\frac{1}{1+\frac{r^\alpha}{sP}}r\mathrm{d}r$.

Proof. See Appendix C. ■

As demonstrated analytically in the next subsection and numerically later in the sequel, both the upper and lower bounds derived above are surprisingly tight.

C. Ratio of the Bounds

To study the tightness of the proposed upper and lower bounds, we derive a tight approximation on the ratio of upper bound and lower bound and show that it is close to one.

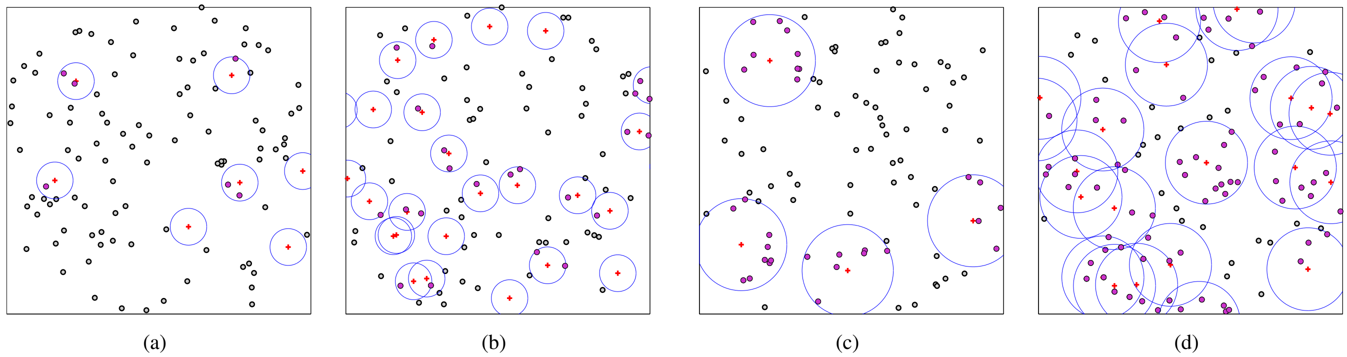

Fig. 2. Four configurations in the PHP network model (a) First: LD-SH, (b) Second: HD-SH, (c) Third: LD-LH, and (d) Fourth: HD-LH.

Proposition 1. *The ratio of the upper and lower bounds on the Laplace transforms derived in Theorems 2 and 1 is*

$$\frac{\mathcal{L}_{\rm u}(s)}{\mathcal{L}_{\rm l}(s)} \approx \int_D^{\infty} \exp\left[-2\pi\lambda_1 \int_{v_1}^{\infty} \left(1-\exp\left(f(v)\right)\right) v{\rm d}v\right] f_{V_1}(v_1){\rm d}v_1, \quad (10)$$

where $\mathcal{L}_{\rm u}(s)$ and $\mathcal{L}_{\rm l}(s)$ denote the proposed upper and lower bounds, given by Theorem 2 and Theorem 1, respectively. Further, $f(v) = 2\lambda_2 \int_{v-D}^{v+D} \arccos\left(\frac{r^2+v^2-D^2}{2vr}\right) \frac{1}{1+\frac{r^{\alpha}}{sP}} r{\rm d}r$.

Proof. See Appendix D. ■

This approximation can be interpreted as the Laplace transform of interference power *removed* by all the holes except the closest hole from the homogeneous PPP Φ_2 after ignoring the effect of overlaps. This will be shown to be tight and close to one across wide range of parameters in the next subsection.

D. Discussion and Numerical Results

In this section, we validate the tightness of the bounds derived in the previous two subsections and compare them with the known results. Simulations are performed over circular region with radius $40m$ and results are averaged over 5×10^4 iterations. Unless otherwise specified, we set the network parameters as follows: $\lambda_2 = 1$, $\alpha = 4$, $P = 1$, $r_0 = 0.1$. We compare our proposed bounds with two known results discussed in the first paragraph of Section III: (i) the first-order statistic approximation provided in [13], and (ii) the PPP-based bound where Ψ is approximated by Φ_2. As Fig. 2 shows, network parameters are adjusted such that four different configurations are realized. The network configuration depends on the densities λ_1, λ_2 and radius D which are the design parameters of the system. We define the possible configurations as: low density of holes and small holes (LD-SH); high density of holes and small holes (HD-SH); low density of holes and large holes (LD-LH); and high density of holes and large holes (HD-LH). Fig. 3 compares the ratio of proposed upper and lower bounds along with an approximation on this ratio derived in Proposition 1. In addition to validating the tightness of the approximation given by Proposition 1, this result shows that the ratio in all cases of interest is close to one, which demonstrates the tightness of the upper and lower

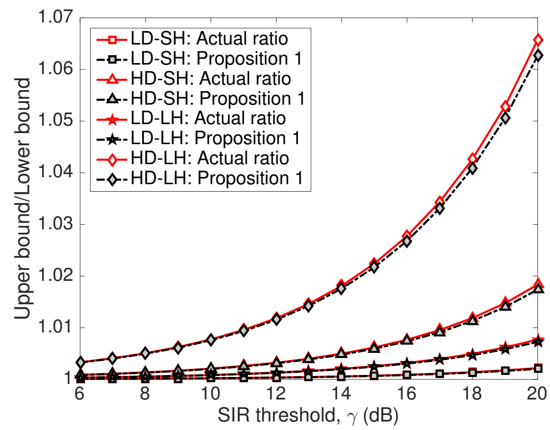

Fig. 3. Ratio of the new upper and lower bounds.

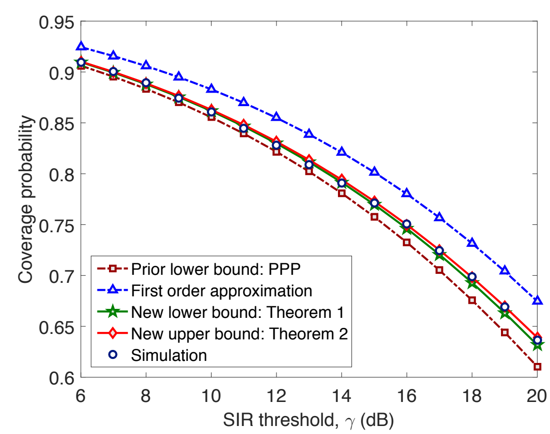

Fig. 4. Analytical and simulation results for the coverage probability of a typical PHP user in a HD-SH scenario ($\lambda_1 = 0.2$ and $D = 0.6$).

bounds derived in Theorems 2 and 1, respectively. Note that, as expected, the ratio is comparatively higher when the holes are large and dense (HD-LH case).

Before going into further details, note that when the holes are small and sparse (LD-SH case), our analytical new lower and upper bounds, the first-order statistic approximation and PPP-based bound are all expected to be fairly tight. This case is at least more benign than the case where holes are small and dense (LD-SH). Similarly, the case where holes are large

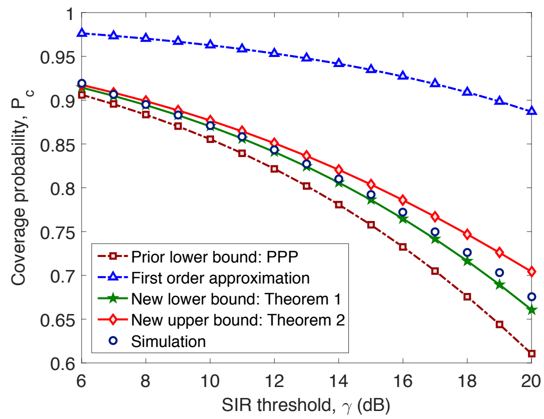

Fig. 5. Analytical and simulation results for the coverage probability of a typical PHP user in a HD-LH scenario ($\lambda_1 = 0.2$ and $D = 1.5$).

and sparse (LD-LH) is more benign than the case where the holes are both large and dense (HD-LH). Because of the space constraints, we focus only on the cases where the holes are dense (HD-SH and HD-LH). If we are able to demonstrate the tightness of bounds in these cases, that will directly imply tightness in the other two cases as well.

We now compare the proposed bounds with the numerical results and the known approaches in terms of coverage probability in Figs. 4 (HD-SH case) and 5 (HD-LH case). As demonstrated in (5), the coverage probability for our setup is simply the Laplace transform of interference evaluated at $s = \frac{\gamma r_0^\alpha}{P}$. In Fig. 4 (HD-SH case), we notice that both the proposed upper and lower bounds provide remarkably accurate characterization of coverage probability. Both the known results provide loose but still reasonable characterization. In Fig. 5, we focus on the worst configuration that could possibly happen (HD-LH). The new bounds again provide an accurate characterization of the coverage probability. On the other hand, both the known results are fairly loose. In particular, the first order statistic approximation does not seem to work at all. This is because by removing points independently from Φ_2, the local neighborhood of the typical point is disturbed leading to a very loose result. On a similar note, the proposed results work fairly well in this case as well because the local neighborhood is preserved while deriving both the bounds.

IV. Conclusion

In this paper, we have provided new easy-to-use provable bounds on the Laplace transform of interference experienced by a typical user in a PHP. In addition to accurately characterizing the interference, these bounds immediately characterize the coverage probability of a typical user in the Rayleigh fading case. Since the prior work has mostly focused on reducing the PHP to a PPP either by ignoring the holes or by matching the PPP density to that of a PHP, to the best of our knowledge, the proposed bounds are the tightest known bounds for the Laplace transform of interference in a PHP. For the analysis, we proposed a new approach in which the holes are *dissolved* in such a way that a PHP is reduced to an equivalent (and more tractable) non-homogeneous PPP. The key in deriving tight bounds was to preserve the *local neighborhood* around the typical point while simplifying the far field to attain tractability. The tightness of the bounds is demonstrated analytically as well as numerically by comparing with simulations and known approaches. These results have numerous applications in a variety of wireless networks where interference management is performed by spatially separating the active links, such as in cognitive radio and D2D networks.

Appendix

A. Proof of Lemma 1

The Laplace transform of interference conditioned on the distance of the hole center to the origin, $\|\mathbf{y}\|$, is

$$\mathcal{L}_{I|\|\mathbf{y}\|}(s) = \mathbb{E}\left[\exp\left(-s\sum_{\mathbf{x}\in\Phi_2\cap\mathbf{b}^c(\mathbf{y},D)} Ph_{\mathbf{x}}\|\mathbf{x}\|^{-\alpha}\right)\right]$$
$$\stackrel{(a)}{=} \exp\left(-\lambda_2\left(\int_{\mathbb{R}^2}\frac{1}{1+\frac{\|\mathbf{x}\|^\alpha}{sP}}\mathrm{d}\mathbf{x} - \int_{\mathbf{b}(\mathbf{y},D)}\frac{1}{1+\frac{\|\mathbf{x}\|^\alpha}{sP}}\mathrm{d}\mathbf{x}\right)\right)$$
$$\stackrel{(b)}{=} \exp\left(-\pi\lambda_2\frac{(sP)^{2/\alpha}}{\mathrm{sinc}(2/\alpha)}\right)\times$$
$$\exp\left(2\lambda_2\int_{\|\mathbf{y}\|-D}^{\|\mathbf{y}\|+D}\frac{\arccos\left(\frac{r^2+\|\mathbf{y}\|^2-D^2}{2\|\mathbf{y}\|r}\right)}{1+\frac{r^\alpha}{sP}}r\mathrm{d}r\right) \quad (11)$$

where (a) follows from $h_{\mathbf{x}} \sim \exp(1)$ and the expression for the PGFL of a PPP [15]. The first integral in (b) follows from the standard machinery, where the integral is first converted form Cartesian to polar coordinates and the closed form expression is then derived by using the properties of the Gamma function [2, Appendix B]. The second term follows from the cosine-law: $r^2+\|\mathbf{y}\|^2-2r\|\mathbf{y}\|\cos\theta(r) = D^2$ (Fig. 1). By substituting $\lambda(r) = \frac{\lambda_2}{\pi}\arccos\left(\frac{r^2+\|\mathbf{y}\|^2-D^2}{2\|\mathbf{y}\|r}\right)$, the final expression in equation (6), is derived.

B. Proof of Theorem 1

The lower bound on Laplace transform of interference is

$$\mathcal{L}_I(s) \geq \mathbb{E}\left[\exp\left(-s\sum_{\mathbf{x}\in\Phi_2\cap\mathbf{b}^c(\mathbf{y}_1,D)} Ph_{\mathbf{x}}\|\mathbf{x}\|^{-\alpha}\right)\right] =$$
$$\int_D^\infty \mathcal{L}_{I|\|\mathbf{y}_1\|}(s;\lambda,D)f_{V_1}(v_1)\mathrm{d}v_1 \stackrel{(a)}{=} \exp\left(-\pi\lambda_2\frac{(sP)^{2/\alpha}}{\mathrm{sinc}(2/\alpha)}\right)\times$$
$$\int_D^\infty \exp\left(\int_{v_1-D}^{v_1+D} 2\lambda_2\frac{\arccos\left(\frac{r^2+v_1^2-D^2}{2v_1 r}\right)}{1+\frac{r^\alpha}{sP}}r\mathrm{d}r\right)\times$$
$$2\pi\lambda_1 v_1\exp(-\pi\lambda_1(v_1^2-D^2))\mathrm{d}v_1 = \exp\left(-\pi\lambda_2\frac{(sP)^{2/\alpha}}{\mathrm{sinc}(2/\alpha)}\right)$$
$$\times\int_D^\infty \exp\left(g(v_1)\right)2\pi\lambda_1 v_1\exp(-\pi\lambda_1(v_1^2-D^2))\mathrm{d}v_1 \quad (12)$$

where $\mathbf{b}(\mathbf{y}_1, D)$ denotes the hole centered at $\mathbf{y}_1$ with radius D, and (a) follows by substituting the conditional Laplace transform expression from Lemma 1, and the PDF of V_1 from (7). Further, $g(v_1) = \int_{v_1-D}^{v_1+D}\arccos\left(\frac{r^2+v_1^2-D^2}{2v_1 r}\right)\frac{2\lambda_2}{1+\frac{r^\alpha}{Ps}}r\mathrm{d}r$.

C. Proof of Theorem 2

By definition, the Laplace transform of the PHP is

$$\mathcal{L}_I(s) \stackrel{(a)}{=} \mathbb{E}\left[\exp\left(-s\sum_{\mathbf{x}\in\Phi_2\cap\Xi_D^c} Ph_{\mathbf{x}}\|\mathbf{x}\|^{-\alpha}\right)\right]$$

$$\stackrel{(b)}{=} \mathbb{E}_{\Phi_1}\left[\exp\left(-\lambda_2\left(\int_{\mathbb{R}^2}\frac{\mathrm{d}\mathbf{x}}{1+\frac{\|\mathbf{x}\|^\alpha}{sP}} - \int_{\Xi_D}\frac{\mathrm{d}\mathbf{x}}{1+\frac{\|\mathbf{x}\|^\alpha}{sP}}\right)\right)\right]$$

where Ξ_D in (a) is $\triangleq \bigcup_{\mathbf{y}\in\Phi_1}\mathbf{b}(\mathbf{y},D)$ as defined in (1), (b) follows by taking expectations over channel gains $h_{\mathbf{x}} \sim \exp(1)$ and the PPP Φ_2 given Ξ_D, where we use the PGFL of a PPP to take expectation over Φ_2. Note the integral over Ξ_D is not easy to compute due to the possible overlaps in the holes. Therefore, to derive the bound, we use

$$\int_{\Xi_D}\frac{\mathrm{d}\mathbf{x}}{1+\frac{\|\mathbf{x}\|^\alpha}{sP}} \leq \sum_{\mathbf{y}\in\Phi_1}\int_{\mathbf{b}(\mathbf{y},D)}\frac{\mathrm{d}\mathbf{x}}{1+\frac{\|\mathbf{x}\|^\alpha}{sP}},$$

which follows by ignoring the effect of overlaps. Substituting this back in the expression of $\mathcal{L}_I(s)$; solving the first integral as done in Lemma 1; and using the result of Lemma 1 to handle the integral over $\mathbf{b}(\mathbf{y},D)$, we get

$$\mathcal{L}_I(s) \leq \exp\left(-\pi\lambda_2\frac{(sP)^{2/\alpha}}{\mathrm{sinc}(2/\alpha)}\right)\times$$

$$\mathbb{E}_{\Phi_1}\left[\prod_{\mathbf{y}\in\Phi_1}\exp\left(2\lambda_2\int_{\|\mathbf{y}\|-D}^{\|\mathbf{y}\|+D}\frac{\arccos\left(\frac{r^2+\|\mathbf{y}\|^2-D^2}{2\|\mathbf{y}\|r}\right)}{1+\frac{r^\alpha}{sP}}r\mathrm{d}r\right)\right] \stackrel{(a)}{=}$$

$$\exp\left(\frac{-\pi\lambda_2(sP)^{2/\alpha}}{\mathrm{sinc}(2/\alpha)}\right)\times\exp\left[-2\pi\lambda_1\int_D^\infty(1-\exp(f(v)))\,v\mathrm{d}v\right],$$

where the second term in (a) follows from the PGFL of a PPP, and then by substituting $\|\mathbf{y}\| = v$ and $f(v) = 2\lambda_2\int_{v-D}^{v+D}\frac{\arccos(\frac{r^2+v^2-D^2}{2vr})}{1+r^\alpha/s}r\mathrm{d}r$. Since by definition of the typical point in this case there are no points of Φ_1 in $\mathbf{b}(0,D)$, the lower bound of integral in the above expression is D.

D. Proof of Proposition 1

Denote the interference powers used for deriving the lower and upper bounds on the Laplace transform of interference in Theorems 1 and 2 by I_{l} and I_{u}, respectively. For instance, $I_{\mathrm{l}} = \sum_{\mathbf{x}\in\Phi_2\cap\mathbf{b}^c(\mathbf{y}_1,D)} Ph_{\mathbf{x}}\|\mathbf{x}\|^{-\alpha}$, where $\mathbf{y}_1$ is the location of the closest point of Φ_1 to the origin. Using this notation, the ratio of the upper and lower bounds is

$$\frac{\mathcal{L}_{\mathrm{u}}(s)}{\mathcal{L}_{\mathrm{l}}(s)} = \frac{\mathbb{E}\left[e^{-sI_{\mathrm{u}}}\right]}{\mathbb{E}\left[e^{-sI_{\mathrm{l}}}\right]} \stackrel{(a)}{\leq} \mathbb{E}\left[e^{-sI_{\mathrm{u}}}\right]\mathbb{E}\left[\frac{1}{e^{-sI_{\mathrm{l}}}}\right] \stackrel{(b)}{\approx} \mathbb{E}\left[e^{-s(I_{\mathrm{u}}-I_{\mathrm{l}})}\right],$$

where (a) follows from the Jensen's inequality, and (b) is an approximation because I_{u} and I_{l} are not truly independent. We will numerically show that the resulting expression provides a tight approximation. As it is clear from the proof of Theorem 2, I_{u} is the effective interference from Φ_2 when holes corresponding to Φ_1 are carved out individually without worrying about the overlaps. In other words, some points of Φ_2 may be *virtually* removed multiple times, thus leading to an upper bound on the Laplace transform. This means $I_{\mathrm{u}} - I_{\mathrm{l}}$ term in the above expression can be interpreted as the effective *interference power* removed by all the holes except the closest hole from the homogeneous PPP Φ_2, where again the overlap among the holes is ignored. On the same lines as the proof of Theorem 2, the term $\mathbb{E}\left[e^{-s(I_{\mathrm{u}}-I_{\mathrm{l}})}\right]$ can be evaluated as

$$\mathbb{E}_{\Phi_1|V_1}\exp\left(2\lambda_2\left(\sum_{\mathbf{y}\in\Phi_1/\mathbf{y}_1}\int_{\|\mathbf{y}\|-D}^{\|\mathbf{y}\|+D}\frac{\arccos(\frac{r^2+\|\mathbf{y}\|^2-D^2}{2\|\mathbf{y}\|r})}{1+r^\alpha/s}r\mathrm{d}r\right)\right)$$

$$\stackrel{(b)}{=} \exp\left[-2\pi\lambda_1\int_{v_1}^\infty(1-\exp(f(v)))\,v\mathrm{d}v\right],$$

where (b) is obtained from PGFL of a PPP and $f(v) = 2\lambda_2\int_{v-D}^{v+D}\arccos\left(\frac{r^2+v^2-D^2}{2vr}\right)\frac{1}{1+\frac{r^\alpha}{Ps}}r\mathrm{d}r$. Note that $V_1 = \|\mathbf{y}_1\|$ is the distance of the closest point of Φ_1 from the origin. Deconditioning over the distance V_1 using the distribution given by (7) completes the proof.

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