Downlink Coverage Probability of K-Tier HetNets with General Non-Uniform User Distributions

Chiranjib Saha and Harpreet S. Dhillon

Abstract—**Current approaches to the analysis of heterogeneous cellular networks (HetNets) with random spatial models assume users to be distributed according to a homogeneous Poisson Point Process (PPP) independently of the base station (BS) locations. In reality, however, current deployments are capacity-driven, which correlates the BS and user locations. In this paper, we develop tools for the downlink analysis of HetNets with general non-uniform user distributions by enriching the K-tier PPP HetNet model. Instead of being PPP distributed, the user locations are modeled by a Poisson cluster process with the cluster centers being the BSs. In particular, we provide the first formal analysis of the downlink coverage probability in terms of a general density function describing the locations of users around the BSs. All the results are specialized to a particular case of a Thomas cluster process, where the locations of the users around BSs are Gaussian distributed. Our results concretely demonstrate that the coverage probability decreases with the increasing variance of the user location distribution, ultimately collapsing to the result for the PPP user distribution when the variance goes to infinity.**

Index Terms—**Non-uniform user distribution, coverage probability, Poisson cluster process, Thomas cluster process, stochastic geometry.**

I. Introduction

Increasing data traffic over mobile networks has necessitated the need for increasing cellular network capacity at an unprecedented rate. Not surprisingly, a key enabler for increasing network capacity at such a rate is a more aggressive frequency reuse. This is already underway in the form of capacity-driven deployment of several types of low-power BSs in the areas of high user density, such as coffee shops, airport terminals, and downtowns of large cities [1]. Due to the coexistence of various types of low-power BSs, collectively called *small cells*, with the conventional macrocells, the resulting network is often termed as a heterogeneous cellular network (HetNet). As a result of the increasing irregularity of BS locations in HetNets, random spatial models have become a preferred choice for the accurate modeling and tractable analysis of these networks. The first comprehensive model for their analysis was proposed in [1] where the locations of different classes of BSs were modeled by independent PPPs and the downlink analysis was performed at a typical user chosen independently of the BS locations. While there has been a significant body of follow-up works since then, quite remarkably, none of them has focused on developing tools for the more realistic case in which the user and BS locations are correlated. In this paper, we fill this gap and provide the first comprehensive downlink analysis of HetNets with non-uniform user distributions in which the BS and user locations may be correlated.

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Related work. As noted above, almost all the prior work in the analysis of HetNets using random spatial models focuses on the case of uniform user distributions. Please refer to [2] for a detailed survey. The existing *albeit* sparse work on the analysis of non-uniform user distributions can be classified into two main directions. The first is to characterize the performance through detailed system-level simulations. As expected, the general philosophy is to capture the capacity-centric deployments by assuming higher user densities in the vicinity of small cell BSs, e.g., see [3]. On similar lines, [4] has introduced a low complexity PPP simulation approach for HetNets with correlated user and BS locations. The correlation coefficient is defined in terms of a "potential function" over Voronoi cells. The second direction, in which the contributions are even more sparser, is to use analytic tools from stochastic geometry to characterize the performance of HetNets with non-uniform user distributions. One notable contribution in this direction is the generative model that we proposed in [5], where non-uniform user distribution is generated from the homogeneous PPP by thinning the BS field independently, conditional on the active link from a typical device to its serving BS. While the resulting model is tractable, it suffers from two shortcomings: (i) it is restricted to single-tier networks and extension to HetNet is not straightforward, and (ii) even for single-tier networks, it does not allow the inclusion of any general non-uniform distribution of users in the model. Besides this, some other attempts have been made at including non-uniform user distributions using simple models, especially in the context of indoor communications, e.g., see [6]. There is thus a need for developing mathematical tools for the analysis of HetNets with non-uniform user distributions. Developing these tools is the main focus of this paper.

Contributions and outcomes. In this paper, we enrich the K-tier PPP HetNet model proposed in [1] to allow for correlation in the BS and user locations. In particular, the user locations are modeled by a Poisson cluster process [7] with the cluster centers being the BSs. Using new distance distribution results for cluster processes that we derived in [8], [9], we derive exact expression for the coverage probability of a typical device in this setup as a function of the distribution of its location with respect to the center of the cluster to which it belongs. Association probabilities of the typical user with the BSs belonging to open access tiers along with the BS at its own cluster center are also derived. The results are then specialized to the case of a Thomas cluster process in which the users are Gaussian distributed around BSs. Furthermore, our results concretely demonstrate that the coverage probability decreases with the increasing variance of the user location distribution, ultimately collapsing to the result for the PPP user distribution when the variance goes to infinity.

II. System Model

Consider a K-tier heterogeneous cellular network, where BSs across tiers (or classes) differ in terms of their transmit powers and deployment densities. For notational simplicity, define $\mathcal{K}=\{1,2,\dots,K\}$ as the indices of the K tiers. The locations of the i^{th}-tier BSs are modeled by an independent homogeneous PPP $\Phi_i^{(\mathrm{BS})}$ of density $\lambda_i^{(\mathrm{BS})}>0$. The i^{th}-tier BSs are assumed to transmit at the same power P_i. As is usually the case, we assume that a fraction of i^{th}-tier BSs are in open access for the user of interest and the rest are in closed access. The i^{th}-tier open and closed access BSs are modeled by two independent PPPs Φ_i and Φ_i' with densities λ_i and λ_i', respectively, where $\Phi_i^{(\mathrm{BS})}=\Phi_i\cup\Phi_i'$ and $\lambda_i^{(\mathrm{BS})}=\lambda_i+\lambda_i'$.

Unlike prior work that focused almost entirely on the performance analysis of users that are uniformly distributed in the network, we focus on the users that are more likely to lie closer to the BSs, especially the small cells. For concreteness, we assume that $\mathcal{B}\subseteq\mathcal{K}$ tiers have clusters of users around their BSs. In particular, given the location of a BS in the i^{th} tier as $\mathbf{x}_i\in\Phi_i$, the users are assumed to be symmetrically, independently, and identically distributed (i.i.d.) around $\mathbf{x}_i$. Union of all such locations of users forms a *Poisson cluster process* of users with respect to the BSs in i^{th} tier, denoted by Φ_i^u, with Φ_i being the parent point process of Φ_i^u. In other words, points of Φ_i serve as cluster centers for Φ_i^u. To maintain generality, we assume that the user location $\mathbf{z}\in\mathbb{R}^2$ with respect to its cluster center at $\mathbf{x}_i$ follows a general distribution with probability density function (PDF) $f^{(i)}(\mathbf{z})$, which may not necessarily be the same across tiers. This allows to capture the fact that depending upon the range of the BSs, users may be densely distributed in small coverage regions of certain classes of BSs, such as femtocells, and more sparsely distributed around certain other classes of BSs, such as picocells. After deriving all the results in terms of the general distributions, we will specialize them for a special case of interest where Φ_i^u is modeled as a *Thomas cluster process* in which the users are scattered according to a symmetric normal distribution of variance σ_i^2 around the BS-s of Φ_i. In this case, we have $f^{(i)}(\mathbf{z})=\frac{1}{2\pi\sigma_i^2}\exp\left(-\frac{\|\mathbf{z}\|^2}{2\sigma_i^2}\right)$, $\mathbf{z}\in\mathbb{R}^2$ [10].

Without loss of generality, downlink analysis is performed at a *typical user* of Φ_i^u, which is a randomly chosen user from a randomly chosen cluster of Φ_i^u, also termed *representative cluster*. Since the PPP-s are stationary, we can transform the origin to the location of this typical user. Now, given that user is at origin, let the location of the representative cluster center is $\mathbf{y}\in\Phi_i$. We can exclude this $\mathbf{y}$ from Φ_i and Slivnyak's theorem guarantees that the remaining process $\Phi_i\setminus\{\mathbf{y}\}$ has the same distribution as Φ_i [10]. For notational simplicity, let us form another tier Φ_0 which consists of only a single point which is $\mathbf{y}$, i.e., $\Phi_0=\{\mathbf{y}\}$. Then, set of indices of all tiers is enriched to $\mathcal{K}_1=\{0\}\cup\mathcal{K}=\{0,1,2,\dots,K\}$. The user can either connect to its own cluster center i.e. the BS in Φ_0, or, to some other BS $\in\Phi_j$ with $j\in\mathcal{K}$. The received power at the location of the typical user at origin from an BS at $\mathbf{x}_k\in\Phi_k$ can be expressed as, $P(\mathbf{x}_k)=P_k h_k\|\mathbf{x}_k\|^{-\alpha}$, where, α is the path loss exponent and h_k is the random channel gain. Under Rayleigh fading assumption, h_k-s are i.i.d. exponential random variables (RV-s) with $h_k\sim\exp(1)$. Due to lack of space, we are delegating the inclusion of shadowing to the extended version of this paper, which will be done using displacement theorem on the same lines as done for downlink cellular analysis under the assumption of uniform user distribution in the literature, e.g., see [11]. We assume average power-based cell selection in which a typical user connects to the BS that provides maximum received power averaged over fading. Note that since the average power from an BS at $\mathbf{x}_j\in\Phi_j$ is $P_j\|\mathbf{x}_j\|^{-\alpha}$, the set of candidate serving BSs is the set of closest BSs from each tier $j\in\mathcal{K}_1$ [11].

III. Association Probability

This is the first technical section of the paper, where we derive the probability that a typical user is served by a given tier $j\in\mathcal{K}_1$, which is usually termed as the *association probability*. We will then derive the distribution of the distance from the typical user to its serving BS conditioned on the serving BS being from a particular tier. To begin the discussion, let R_j be the RV denoting the distance from the typical user at the origin to the nearest point of Φ_j. Since Φ_j ($j\in\mathcal{K}$) are independent homogeneous PPP-s, the distribution of R_j, $j\in\mathcal{K}$, is [10]

$$\text{PDF:}\qquad f_{R_j}(r)=2\pi\lambda_j\exp(-\pi\lambda_j r^2)r,\ r\geq 0,\qquad(1a)$$

$$\text{CCDF:}\qquad \overline{F}_{R_j}(r)=\exp(-\pi\lambda_j r^2),\ r\geq 0.\qquad(1b)$$

However, the distribution of R_j will be different for $j=0$ as Φ_0 contains only a single point with predefined distance distribution. With user at origin, the distribution of the position "vector" of the cluster center at $\mathbf{x}_0$ will have the same i.i.d. distribution as $f^{(i)}(\mathbf{x}_0)$. Equivalently, the position can be characterized in terms of polar coordinates (R_0,Θ_0) with joint distribution $f_{R_0,\Theta_0}(r,\theta)$. The marginal distribution of distance R_0 can now be computed by integrating over $f_{R_0,\Theta_0}(r,\theta)$ as

$$f_{R_0}(r)=\int_0^{2\pi} f_{R,\Theta}(r,\theta)\mathrm{d}\theta.\qquad(2)$$

In the special case when Φ_i^u is a Thomas cluster process, R_0 is Rayleigh distributed with PDF and CCDF [8], [9]

$$\text{PDF:}\qquad f_{R_0}(r)=\frac{r}{\sigma_i^2}e^{\frac{-r^2}{2\sigma_i^2}},\ r\geq 0,\qquad(3a)$$

$$\text{CCDF:}\qquad \overline{F}_{R_0}(r)=e^{\frac{-r^2}{2\sigma_i^2}},\ r\geq 0.\qquad(3b)$$

To derive association probability, let S_{Φ_j} be the event that a typical user is served by a BS from the j^{th} tier. Thus, S_{Φ_j} can be defined in terms of its indicator function as $\mathbf{1}_{S_{\Phi_j}}=$

$$\mathbf{1}(\arg\max_{j\in\mathcal{K}_1} P_j R_j^{-\alpha}=j)=\bigcap_{k\in\mathcal{K}_1}\mathbf{1}\left(R_k>\bar{P}_{jk}R_j\right),\qquad(4)$$

where $\bar{P}_{jk}=\left(\frac{P_k}{P_j}\right)^{1/\alpha}$ and $\mathbf{1}(.)$ is the indicator function of the random vector $\mathbf{R}=[R_0,R_1,...,R_k]$. Note that since the

0^{th} tier is derived from the i^{th} tier, $P_0 \equiv P_i$. The association probability for each tier is now defined as follows.

Definition 1. *Association Probability, $\mathcal{A}_j$ for j^{th} tier, $\forall j \in \mathcal{K}_1$ is defined as the probability that the typical user will be served by the j^{th} tier. Mathematically, it can be expressed as*

$$\mathcal{A}_j = \mathbb{P}(S_{\Phi_j}). \tag{5}$$

The following lemma provides general expression for $\mathcal{A}_j$. The proof is provided in Appendix A.

Lemma 1. *Association probability of the j^{th} tier is*

$$\mathcal{A}_j = \int_{r>0} \prod_{k\in\mathcal{K}_1} \overline{F}_{R_k}\left(\bar{P}_{jk} r\right) f_{R_j}(r)\mathrm{d}r. \tag{6}$$

Now we derive the distribution of the distance from the typical user to its serving BS in Φ_j, located at $\mathbf{x}_j \in \Phi_j$. We denote this distance by $\mathcal{X}_j = \|\mathbf{x}_j\|$. Conditioned on the association with the j^{th} tier, this serving distance is simply the distance to the nearest BS in Φ_j. Hence $\mathcal{X}_j$ is related to R_j as $\mathcal{X}_j = R_j|S_{\Phi_j}$. The PDF of $\mathcal{X}_j$ is derived in the next Lemma. The proof is provided in Appendix B.

Lemma 2. *The PDF of $\mathcal{X}_j$, i.e. the distance between a typical user and its serving BS in j^{th} tier is given by*

$$f_{\mathcal{X}_j}(x) = \frac{1}{\mathcal{A}_j} \prod_{k\in\mathcal{K}_1\setminus\{j\}} \overline{F}_{R_k}(\bar{P}_{jk}x) f_{R_j}(x). \tag{7}$$

With the general distance distribution derived above, let us now specialize it for the case when Φ_i^u is a Thomas cluster process. For a cleaner exposition, we will enrich the notation to include $j = 0$ in the following Corollary. The proof is provided in Appendix C.

Corollary 1. *If Φ_i^u is Thomas cluster process, the serving distance distribution from j^{th} tier BS, $j \in \mathcal{K}_1$, is*

$$f_{\mathcal{X}_j}(x) = \frac{2\pi\lambda_j}{\mathcal{A}_j} \exp\left(-\pi \sum_{k=0}^{K} \lambda_k (\bar{P}_{jk}x)^2\right) x, \tag{8}$$

where $\mathcal{A}_j$ is the association probability with the j^{th} tier derived for the general case in Lemma 1. For the Thomas cluster process, it reduces to

$$\mathcal{A}_j = \frac{\lambda_j}{\sum\limits_{k=0}^{K} \bar{P}_{jk}^2 \lambda_k},\ \forall j \in \mathcal{K}_1, \tag{9}$$

where λ_0 is defined as $\lambda_0 = \frac{1}{2\pi\sigma_i^2}$.

IV. Coverage Probability Analysis

This is the second technical section of the paper where we use the association probability and the distance distribution results derived in the previous section to derive easy-to-use expressions for the coverage probability of a typical user of Φ_i^u. From the above discussion, it is easy to deduce that if the typical user is served by a BS $\in \Phi_j$ located at a distance $\|\mathbf{x}_j\| = \mathcal{X}_j$ from the typical user, there exists no k^{th} tier BSs, $\forall k \in \mathcal{K}_1$, within a disc of radius, $x_{jk} = \bar{P}_{jk}x$ centered at the user, where x is an instance of RV $\mathcal{X}_j$. We denote this disc by $b(\mathbf{0}, x_{jk})$. Assuming association with j^{th} tier, the total interference experienced by the typical user originates from two independent sets of BSs: (i) $\cup_{k\in\mathcal{K}_1}\Phi_k \setminus \mathbf{x}_j$, the set of open access BSs excluding the serving BS and (ii) $\cup_{k\in\mathcal{K}}\Phi'_k$, the set of closed access BSs. As all the interferers from the k^{th} open access tier will lie outside $b(\mathbf{0}, x_{jk})$, we define interference from k^{th} open-access tier as, $\mathcal{I}_{o(j,k)}(\|\mathbf{x}_j\|) = \sum_{\mathbf{x}_k\in\Phi_k\setminus\mathbf{x}_j} P_k h_k \|\mathbf{x}_k\|^{-\alpha} = \sum_{\mathbf{x}_k\in\Phi_k\setminus b(\mathbf{0},x_{jk})} P_k h_k \|\mathbf{x}_k\|^{-\alpha}$. The contribution of interference from all open access tiers is:

$$\mathcal{I}_{o(j)}(\|\mathbf{x}_j\|) = \sum_{k=0}^{K} \mathcal{I}_{o(j,k)}(\|\mathbf{x}_j\|). \tag{10}$$

While it is clear that the interference from the open-access tiers defined above depends on the serving distance $\|\mathbf{x}_j\|$, it is not the case with the closed access tiers. Recall that since the closed access tiers do not participate in the cell selection procedure, there is no exclusion zone in their interference field. In particular, the closed access BSs may lie closer to the typical user than its serving BS. We denote the closed access interference by $\mathcal{I}_c = \sum_{m=1}^{K} \mathcal{I}_{cm}$, where $\mathcal{I}_{cm}$ is the interference from all the BSs of the m^{th} closed access tier Φ'_m. Using the notation defined above, we can now define signal-to-interference-ratio ($\mathtt{SIR}$) at the typical receiver when it is served by the BS located at a distance $\|\mathbf{x}_j\|$ as

$$\mathtt{SIR}(\|\mathbf{x}_j\|) = \frac{P_j h_j \|\mathbf{x}_j\|^{-\alpha}}{\sum\limits_{k=0}^{K} \mathcal{I}_{o(j)}(\|\mathbf{x}_j\|) + \mathcal{I}_c}. \tag{11}$$

Now, we introduce the notion of *coverage* by tier j as the event, $\mathtt{SIR}(\mathcal{X}_j) > \tau$, where τ denotes modulation-coding specific $\mathtt{SIR}$ threshold required for successful reception. Thus the coverage probability can be formally defined as follows.

Definition 2 (Per-tier coverage probability)**.** *Define the per-tier coverage probability as the probability that the typical user of Φ_i^u is in coverage conditioned on the fact that it is served by a j^{th} tier BS. Mathematically,*

$$\begin{aligned}\mathtt{P}_{\mathtt{c}j} &= \mathbb{P}(\mathtt{SIR}(\|\mathbf{x}_j\|)) > \tau | S_{\Phi_j}) = \mathbb{P}(\mathtt{SIR}(\mathcal{X}_j) > \tau) \\ &= \int_{x>0} \mathbb{P}(\mathtt{SIR}(x) > \tau) f_{\mathcal{X}_j}(x)\mathrm{d}x. \end{aligned} \tag{12}$$

The overall coverage probability can now be defined in terms of the per-tier coverage probability as:

$$\mathtt{P}_\mathtt{c} = \sum_{j\in\mathcal{K}_1} \mathbb{P}(S_{\Phi_j})\mathbb{P}(\mathtt{SIR}_j(x) > \tau | S_{\Phi_j}) = \sum_{j\in\mathcal{K}_1} \mathcal{A}_j \mathtt{P}_{\mathtt{c}j}. \tag{13}$$

Recall that the association probability $\mathcal{A}_j$ has already been derived in Lemma 1. We now focus on the derivation of per-tier coverage probability $\mathtt{P}_{\mathtt{c}j}$. Note that using the Rayleigh fading assumption along with the fact that the open access interference terms $\{\mathcal{I}_{o(j,k)}\}$ and the closed access interference terms $\{\mathcal{I}_{cm}\}$ are all independent of each other, we can express the per-tier coverage probability in terms of the product of Laplace transforms of these interference terms. The result is given in the next Theorem. Since this follows from standard arguments, e.g., see [1], [12], we are skipping the proof due to space constraints. Instead, we will focus on the proofs of the Laplace transforms of interference, which are unique to this work due to the non-uniform user distribution assumption.

Theorem 1. *(Per-tier coverage probability) Coverage probability of the typical user from Φ_i^u conditional on the serving BS being from the j^{th} tier is* $\mathtt{P}_{\mathtt{c}j} =$

$$\int\limits_{x>0} \prod_{k=0}^{K} \mathcal{L}_{\mathcal{I}_{o(j,k)}}\left(\tfrac{\tau x^\alpha}{P_i}\right) \prod_{m=1}^{K} \mathcal{L}_{\mathcal{I}_{cm}}\left(\tfrac{\tau x^\alpha}{P_i}\right) f_{\mathcal{X}_j}(x)\mathrm{d}x, \qquad (14)$$

where $\mathcal{L}_{\mathcal{I}_{o(j,k)}}(s) = \mathbb{E}\left[\exp(-s\mathcal{I}_{o(j,k)})\right]$, *and* $\mathcal{L}_{\mathcal{I}_{cm}}(s) = \mathbb{E}\left[\exp(-s\mathcal{I}_{cm})\right]$ *respectively denote the Laplace transforms of open and closed access terms.*

The following three Lemmas deal with the Laplace transforms of the different components of interference. We first focus on the interference originating from all the open access tiers except the 0^{th} tier that will need a separate treatment.

Lemma 3. *Given a typical user of Φ_i^u is served by a BS $\in \Phi_j$, Laplace transform of $\mathcal{I}_{o(j,k)}$, $\forall k \in \mathcal{K}$, evaluated at $\frac{\tau x^\alpha}{P_j}$ is*

$$\mathcal{L}_{\mathcal{I}_{o(j,k)}}\left(\frac{\tau x^\alpha}{P_j}\right) = \exp\left(-\pi \bar{P}_{jk}^2 \lambda_k G x^2\right), \qquad (15)$$

where $G = \frac{2\tau}{\alpha-2} {}_2\mathcal{F}_1\left[1, 1-\frac{2}{\alpha}; 2-\frac{2}{\alpha}, -\tau\right]$ *and* ${}_2\mathcal{F}_1$ *is the Gaussian Hypergeometric function.*

Proof: Please refer to Appendix D for the proof. ■

After dealing with the interference from all open access tiers, except the 0^{th} tier, we now focus on the 0^{th} tier, which consists of only the cluster center. Recall that since the 0^{th} tier is created from Φ_i (tier in which the typical user is located), the channel fading gain h_0, transmit power P_0 and exclusion regions will be the same as those of Φ_i.

Lemma 4. *Given a typical user of Φ_i^u connects to the BS $\in \Phi_j$ with $j \in \mathcal{K}$, Laplace transform of $\mathcal{I}_{o(j,0)}$ can be expressed as,*

$$\mathcal{L}_{\mathcal{I}_{o(j,0)}}(s) = \int_{x_{ji}}^{\infty} \frac{1}{1+sP_i r^{-\alpha}} \frac{f_{R_0}(r)}{\bar{F}_{R_0}(x_{ji})} \mathrm{d}r. \qquad (16)$$

Proof: Due to the formation of virtual exclusion zone around the typical user, the cluster center, acting as an interferer, will lie outside $b(\mathbf{0}, x_{j0})$. Thus the distribution of its distance from the typical user, f_{R_0}, will be conditioned on $R_0 > x_{j0} = x_{ji}$. The conditional PDF of R_0 is $f_{R_0}(x|R_0 > y) = \frac{f_{R_0}(x)}{\bar{F}_{R_0}(y)}$, $x \geq y$. Hence, the Laplace transform is

$$\begin{aligned}
\mathcal{L}_{\mathcal{I}_{o(j,0)}}(s) &= \mathbb{E}\left[\exp\left(-sP_i h_i \|\mathbf{x}_i\|^{-\alpha}\right) | \mathcal{X}_j\right] \\
&\overset{(a)}{=} \mathbb{E}_{R_0}\left[\frac{1}{1+sP_i {R_0}^{-\alpha}} | R_0 > x_{ji}\right] \\
&= \int_{x_{ji}}^{\infty} \frac{1}{1+sP_i r^{-\alpha}} f_{R_0}(r|R_0 > x_{ji})\mathrm{d}r,
\end{aligned}$$

where (a) follows from $h_i \sim \exp(1)$, and the result follows by the conditional PDF of R_0 stated above. ■

Lemma 5. *Given the typical user of Φ_i^u connects to any BS $\in \Phi_j$, $\forall j \in \mathcal{K}_1$, Laplace transform of $\mathcal{I}_{cm}$ is*

$$\mathcal{L}_{\mathcal{I}_{cm}}\left(\frac{\tau x^\alpha}{P_j}\right) = \exp\left(-\pi \lambda_m' H (\bar{P}_{jm} x)^2\right), \qquad (17)$$

where $H = \tau^{2/\alpha} \frac{2\pi \csc(\frac{2\pi}{\alpha})}{\alpha}$.

Proof: The proof follows on the same lines as that of Lemma 3, with the only difference being the fact that $\mathcal{I}_{cm}$ is independent of $\mathcal{X}_j$ and hence, the lower limit of the integral will be zero. Thus, we will obtain,

$$\mathcal{L}_{\mathcal{I}_{cm}} = \exp\left(-2\pi\lambda_m' \int_0^\infty \left(1 - \tfrac{1}{1+sP_m r^{-\alpha}}\right) r\mathrm{d}r\right) \qquad (18)$$

The final form can be obtained by some algebraic manipulations as shown in [1, Appendix B]. ■

The expressions of Laplace transforms of different components of interference derived in the above three Lemmas can now be substituted in Eq. 14 to get the coverage probability expression for the general case in terms of the general distance distribution $f_{R_0}(\cdot)$. For the case of Thomas cluster process, the specialized result is much simpler and is given below.

Corollary 2. *If Φ_i^u is a Thomas cluster process, the per-tier coverage probabilities of a typical user of Φ_i^u are*

$$\mathtt{P}_{\mathtt{c}0} = \frac{\lambda_0}{\mathcal{A}_0 \sum\limits_{k=1}^{K} \left(\lambda_0 + (G+1)\lambda_k + H\lambda_k'\right) \bar{P}_{ik}^2} \qquad (19)$$

$$\begin{aligned}
\mathtt{P}_{\mathtt{c}j} = \frac{2\pi\lambda_j}{\mathcal{A}_j} \int\limits_{x>0} \int\limits_{x_{ji}}^{\infty} \frac{1}{\sigma_i^2} \frac{\exp(-\frac{r^2}{2\sigma_i^2})}{1+\tau\left(\frac{r}{x_{ji}}\right)^{-\alpha}} r\,\mathrm{d}r \\
\times \exp\left[-\sum_{k=1}^{K} \pi\left((G+1)\lambda_k + H\lambda_k'\right)(\bar{P}_{jk}x)^2\right] x\mathrm{d}x.
\end{aligned} \qquad (20)$$

Proof: When the user connects to its own cluster center, the interference components are $\mathcal{I}_{cm}$ and $\mathcal{I}_{o(0,k)}$, $\forall k \in \mathcal{K}$. Thus, in Eq. 14, substituting $\mathcal{L}_{\mathcal{I}_{cm}}$ by Eq. 17 and $\mathcal{L}_{\mathcal{I}_{o(0,k)}}$ by Eq. 16 we obtain $\mathtt{P}_{\mathtt{c}0}$ as above. Here, we put $f_{\mathcal{X}_0}$ from Eq. 8.

For coverage by the j^{th} tier ($j \in \mathcal{K}$), the interference components are $\mathcal{I}_{cm}$, $\mathcal{I}_{o(j,k)}$ and $\mathcal{I}_{o(j,0)}$, $\forall k \in \mathcal{K}_1$. The corresponding terms in Eq. 14 are substituted from Eq. 17, Eq. 15 and 16 respectively. The PDF and CCDF of R_0 for Thomas cluster process is given by Eqs. 3a and 3b. ■

The total coverage probability can now be derived by combining the above result with the association probabilities derived for the Thomas cluster process model in Corollary 1.

We now contrast our model with the classical PPP assumption of users. While we do not have space to go over the proofs, it can be shown from the total coverage probability result that (i) $\mathtt{P}_\mathtt{c}$ decreases as the clusters become sparser (i.e., σ_i^2 increases) and (ii) in the limit that $\sigma_i^2 \to \infty$, $\mathtt{P}_\mathtt{c}$ corresponding to users being modeled as Thomas cluster process monotonically converges to the expressions for uniform PPP of users. As variance of user distribution increases, $\mathtt{P}_{\mathtt{c}0}$ vanishes and $\mathtt{P}_{\mathtt{c}j}$ ($j \in \mathcal{K}$) tends to the form obtained for PPP of users. This fact is stated formally in the following remark.

Remark 1. $\mathtt{P}_\mathtt{c}$ *is a monotonically decreasing function of σ_i^2. Further, the following limit can be established for* $\mathtt{P}_\mathtt{c}$,

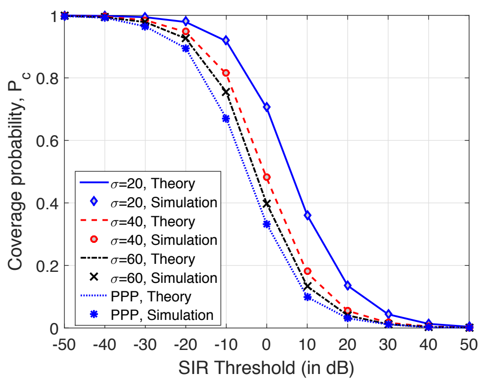

Fig. 1. Coverage probability for *non-uniform* user distribution model when the user locations are sampled from a Thomas cluster process. The baseline *uniform* case when the user distribution is a PPP is also included.

$$\lim_{\sigma_i^2\to\infty} \mathtt{P_c} =$$
$$\sum_{k=1}^{K} 2\pi\lambda_j \int_{x>0} \exp\left(-\sum_{k=1}^{K}\pi((G+1)\lambda_k + H\lambda_k')(\bar{P}_{jk}x)^2\right) x\mathrm{d}x,$$

which is the same result derived for the users modeled as a PPP in a K-tier HetNet in [12]. From Eq. 19, it is pretty straightforward to show that $\lim_{\sigma_i^2\to\infty}\mathtt{P_{c0}} = 0$*. For the other* $\mathtt{P_{c\mathit{j}}}$ $(j \neq 0)$*, the detailed proof is omitted due to the space limitation. But the most crucial part of the proof is the fact that the inner integral in the* $\mathtt{P_{c\mathit{j}}}$ *term given by Eq. 20 converges to a limit, i.e.,* $\int_{x_{ji}}^{\infty} \frac{\exp\left(-\frac{r^2}{2\sigma_i^2}\right)}{\sigma_i^2\left(1+\tau(r/x_{ji})^{-\alpha}\right)} r\mathrm{d}r \to 1$ *as* $\sigma_i^2 \to \infty$.

V. Numerical Results and Discussions

In this section, we validate our key analytical results and provide insights into the network performance under non-uniform user distribution models. Before presenting the results, we briefly delineate the simulation procedure. For concreteness, we restrict our simulation to two tiers: one macrocell tier with density λ_1 with all open access BSs, and one small cell tier with a mix of open and closed access BSs. For the second tier, the open and closed access BS densities are λ_2 and λ_2', respectively. We choose $\lambda_2 = \lambda_2' = 100\lambda_1 = 100$ BSs per $\pi(500)^2$ m^2. The transmit powers are assumed to be $P_1 = 10^3 P_2$. Within a large spatial window of $\pi(4000)^2$ m^2, independent PPPs with corresponding densities are generated. For every realization, a BS in the i^{th} tier is randomly selected and the location of a typical user is generated according to the density function of Thomas cluster process stated in Eq. 3a. In Fig. 1, the coverage probability is plotted for different values of $\mathtt{SIR}$ threshold τ and cluster variance σ_2. First, it is evident that the analytically obtained curves exactly match the curves obtained by simulation. For comparison, the coverage probability assuming homogeneity of users (i.e., PPP assumption) is also plotted. The plots clearly indicate that the coverage probability is significantly higher in the the non-uniform case compared to that in the PPP case. As expected, the coverage in the non-uniform case converges to that in the PPP case as σ_2 increases. With decrease in σ_2, users inside the clusters are more likely to be served by the BS at their cluster centers. This is also evident from the plots of association probabilities as functions of σ_2 in Fig. 2, which clearly show that a user is more likely to be served by its cluster center if the distribution is more "dense" around the cluster center.

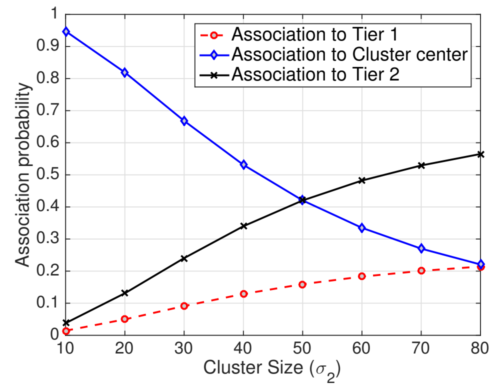

Fig. 2. Association probabilities when Φ_i^u is a Thomas cluster process.

VI. Conclusion

While the random spatial models have been used successfully to study various aspects of HetNets in the past few years, quite remarkably all these works assume the BS and user distributions to be independent. In particular, the analysis is usually performed for a typical user whose location is sampled independently of the BS locations. This is clearly not the case in current capacity-driven deployments where the BSs are deployed in the areas of high user density. This paper presented the first comprehensive analysis of a HetNet where user and BS locations are correlated. In particular, modeling the user locations as a general Poisson cluster process, with BSs being the cluster centers, we have developed new tools leading to tractable results for the downlink coverage probability of a typical user. We have also specialized the results for the case of Thomas cluster process in which the users are Gaussian distributed around BSs. We have also shown that the coverage probability decreases when the clusters become sparser in the Thomas cluster process, ultimately reducing in the limiting case to the coverage derived for the PPP user distribution.

This work opens up a new dimension in the HetNet analysis by providing tools for the analysis of non-uniform user distributions. While there are numerous extensions of this work, immediate followup works could include analyses of downlink rate coverage, biased cell selection and load balancing.

Appendix

A. Proof of Lemma 1

By definition of $\mathcal{A}_j$ in Eq. 5, we have

$$\mathcal{A}_j = \mathbb{E}_{\mathbf{R}}\left[\bigcap_{k\in\mathcal{K}_1\setminus\{j\}} \mathbf{1}\left(R_k > \bar{P}_{jk}R_j\right)\right]$$
$$\stackrel{(a)}{=} \mathbb{E}_{R_j}\prod_{k\in\mathcal{K}_1\setminus\{j\}} \mathbb{P}\left(R_k > \bar{P}_{jk}R_j | R_j\right)$$

$$= \int_{r>0} \prod_{k\in\mathcal{K}_1\setminus\{j\}} \mathbb{P}\left(R_k > \bar{P}_{jk} r\right) f_{R_j}(r)\mathrm{d}r$$

$$\overset{\text{(b)}}{=} \int_{r>0} \prod_{k\in\mathcal{K}_1\setminus\{j\}} \overline{F}_{R_k}(\bar{P}_{jk} r) f_{R_j}(r)\mathrm{d}r,$$

where (a) comes from the fact that Φ_i-s are independent, hence are R_i-s and (b) is directly from the definition of CCDF.

B. Proof of Lemma 2

Let us first derive the CCDF of $\mathcal{X}_j$ below.

$$\mathbb{P}[\mathcal{X}_j > x] = \mathbb{P}[R_j > x | S_{\Phi_j}] = \frac{\mathbb{P}\left(R_j > x, S_{\Phi_j}\right)}{\mathbb{P}(S_{\Phi_i})}$$

$$= \frac{1}{\mathcal{A}_j} \prod_{k\in\mathcal{K}_1\setminus\{j\}} \left[\mathbb{P}(P_j R_j^{-\alpha} > P_k R_k^{-\alpha} | R_j > x)\right]$$

$$= \frac{1}{\mathcal{A}_j} \int_x^\infty \prod_{k\in\mathcal{K}_1\setminus\{j\}} \overline{F}_{R_k}(\bar{P}_{jk} r) f_{R_j}(x)\mathrm{d}x. \tag{21}$$

Hence, the distribution of $\mathcal{X}_j$ is obtained by

$$f_{\mathcal{X}_j}(r) = \frac{\mathrm{d}}{\mathrm{d}x}\left(1 - \mathbb{P}[\mathcal{X}_j > x]\right) = \frac{\prod\limits_{k\in\mathcal{K}\setminus\{j\}} \overline{F}_{R_k}\left(\bar{P}_{jk} x\right) f_{R_j}(x)}{\mathcal{A}_j}.$$

C. Proof of Corollary 1

When $j = 0$, putting f_{R_0} for Thomas cluster process from Eq. 3a and $\overline{F}_{R_k}$ from Eq. 1b in Eq. 6 we have, $\mathcal{A}_0 =$

$$\int\limits_{r>0} \exp\left(-\pi \sum_{k=1}^{K} (\bar{P}_{0k} r)^2\right) \frac{1}{\sigma_i^2} \exp\left(-\frac{r^2}{2\sigma_i^2}\right) r\mathrm{d}r$$

$$= \frac{\frac{1}{2\pi\sigma_i^2}}{\sum\limits_{k=1}^{K} \bar{P}_{0k}^2 \lambda_k + \frac{1}{2\pi\sigma_i^2}}. \tag{22}$$

Putting $\lambda_0 = \frac{1}{2\pi\sigma_i^2}$ as defined, we get the desired result. Now, for $j \in \mathcal{K}$, we substitute $\overline{F}_{R_0}$ from Eq. 1b and f_{R_j} from Eq. 1a, hence, we have, $\mathcal{A}_j =$

$$\int\limits_{r>0} \exp\left(-\pi \sum_{\substack{k=1,\\k\neq j}}^{K} \lambda_k (\bar{P}_{jk} r)^2\right) \exp(-\tfrac{r^2}{2\sigma_i^2}) 2\pi\lambda_j \exp(-\pi\lambda_j r^2) r\mathrm{d}r.$$

Solving the integrals, we find closed form expressions for $\mathcal{A}_j$. The distribution of serving distance when the user is served by its own cluster center is given by putting $j = 0$ in Eq. 7:

$$f_{\mathcal{X}_0} = \frac{1}{\mathcal{A}_0} \prod_{k\in\mathcal{K}} \overline{F}_{R_k}(\bar{P}_{jk} x) f_{R_0}(x)$$

$$\overset{\text{(a)}}{=} \frac{1}{\mathcal{A}_0 \sigma_i^2} \exp\left(-\pi\Big(\sum_{k=1}^{K} \lambda_k \bar{P}_{jk}^2 + \tfrac{1}{2\pi\sigma_i^2}\Big) x^2\right) x, \tag{23}$$

where (a) is obtained by substituting f_{R_0} from Eq.3a and $\overline{F}_{R_k}$ from Eq. 1b. Similarly, for $j \in \mathcal{K}$, substituting $\mathcal{A}_j$ from Corollary 1, f_{R_j} from Eq. 1a, $\overline{F}_{R_k}$ for $k \in \mathcal{K}$ from Eq. 1b and $\overline{F}_{R_0}$ from Eq. 3b, we get,

$$f_{\mathcal{X}_j}(x) = \frac{1}{\mathcal{A}_j} \prod_{k\in\mathcal{K}_1\setminus\{j\}} \exp\left(-\pi\lambda_k (\bar{P}_{jk} x)^2 - \frac{x^2}{2\sigma_i^2}\right)$$

$$= \frac{2\pi\lambda_j}{\mathcal{A}_j} \exp\left(-\pi\left(\sum_{k=1}^{K} \lambda_k \bar{P}_{jk}^2 + \frac{1}{2\pi\sigma_i^2}\right) x^2\right) x.$$

The final expressions follow from simple rearrangements.

D. Proof of Lemma 3

By definition, the Laplace transform of interference is

$$\mathcal{L}_{\mathcal{I}_{o(j,k)}}(s) = \mathbb{E}(\exp(-s\mathcal{I}_{o(j,k)}) | \mathcal{X}_j)$$

$$\overset{\text{(a)}}{=} \mathbb{E}_{\Phi_k}\left[\prod_{\mathbf{x}_k\in\Phi_k\setminus\mathbf{x}_j} \mathbb{E}_{h_k}\left(\exp\left(-sP_k h_k \|\mathbf{x}\|^{-\alpha}\right)\right) | \mathcal{X}_j\right]$$

$$\overset{\text{(b)}}{=} \mathbb{E}_{\Phi_k}\left[\prod_{\mathbf{x}_k\in\Phi_k\setminus\mathbf{x}_j} \frac{1}{1 + sP_k \|\mathbf{x}\|^{-\alpha}} | \mathcal{X}_j\right]$$

$$\overset{\text{(c)}}{=} \exp\bigg(-2\pi\lambda_k \int_{x_{jk}}^{\infty} \left(1 - \frac{1}{1 + sP_k r^{-\alpha}}\right) r.\mathrm{d}r\bigg),$$

where (a) is due to the i.i.d. assumption of h_k, (b) follows from $h_k \sim \exp(1)$, (c) follows from probability generating functional of homogeneous PPP [10]. Putting $s = \frac{\tau x^\alpha}{P_j}$ and after few algebraic manipulations similar to Appendix B of [12], we obtain the desired result.

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