A Simple Direction-Based Lattice Decoding Algorithm for Golden Codes

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Abstract— A quadrant localized lattice decoding algorithm for the golden codes is proposed in this paper that requires very low computational complexity as compared to the sphere decoder. The proposed algorithm is mainly based on the idea of utilizing its directional attributes and localizing the search in a single quadrant of the multi-dimensional space in which the received point lies. We specifically apply this algorithm to the conventional sphere decoder and analyze the performance of the proposed decoder in comparison to the latter one. The adopted approach, however, is sufficiently general to be applied to any popular decoding algorithm for reducing its computational complexity.

I. INTRODUCTION

Multiple-Input Multiple-Output (MIMO) systems have attracted the attention of the communications community due to their huge capacity and improved transmission reliability. The most important contributor to the reliable wireless transmission in the MIMO systems is their ability to provide space diversity and it is important to develop algorithms that take advantage of this fact. Many algorithms with reasonable complexity and performance have been proposed, for example diversity techniques and diversity combining methods. Among them, the most successful one is Space Time Codes (STC) [1]. In STCs, the signal processing at the transmitter is done not only in the time dimension, but also in the spatial dimension. Redundancy is added coherently to both the dimensions. For efficient practical implementation of STCs, the receiver complexity plays a major role. Among the most popular techniques, maximum likelihood (ML) decoding yields the best performance, but is often considered to be practically infeasible due to high computational complexity in MIMO systems with large number of antennas and high-order constellations.

To lower the complexity, the so-called lattice theory is applied to encode and decode the signals in the multiple-antenna digital transmission systems. The most popular lattice decoding algorithm, also termed as sphere decoding algorithm, has near ML performance with reasonably low complexity. It is based on the Fincke and Pohst's strategy [2]. However, the approach has limited applications in practical MIMO systems due to its cubic complexity [3]. Many attempts have been made thereafter to reduce the complexity of the lattice decoders further. In one such attempt [4], it has been shown that Schnorr-Euchner strategy is computationally more efficient than the conventional Pohst algorithm. Some other techniques, such as thresholding [5], exploits the tree search features of the conventional sphere decoder to reduce its complexity. Despite these efforts, the decoder complexity still remains to be the bottle-neck for the implementation of the MIMO systems.

In this paper, we propose a new algorithm, which exploits the directional attributes of the received points to localize the search in a single quadrant of the multi-dimensional space and hence reduces the computational complexity of the existing lattice decoding algorithms significantly. We specifically apply this algorithm to the conventional sphere decoder and analyze the performance of the proposed decoder in comparison to the latter one. The adopted approach, however, is sufficiently general to be applied to any popular decoding algorithm for reducing its computational complexity. Note that a full-rate STC, i.e., the golden code, is used in the encoder of the present system to exploit its property of being an optimal code in the Rayleigh fading environment.

II. THE GOLDEN CODE

The encoder of the present system employs golden code, which is a space-time code for 2×2 coherent MIMO systems. It is a full rate space-time code with non-vanishing determinant [6]. It encodes four QAM symbols (a, b, c and d) at a time to be transmitted from two different antennas in two successive intervals. The codeword $\mathbf{X}$ of the golden code is 2×2 complex matrix of the following form:

$$\mathbf{X}(a,b,c,d) = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha(a+b\theta) & \alpha(c+d\theta) \\ \bar{\alpha}(\gamma(c+d\bar{\theta})) & \bar{\alpha}(a+b\bar{\theta}) \end{bmatrix}, \quad (1)$$

where $\theta = (1+\sqrt{5})/2; \bar{\theta} = (1-\sqrt{5})/2; \alpha = 1 + j(1-\theta) = 1 + j\bar{\theta}; \bar{\alpha} = 1 + j(1-\bar{\theta}) = 1 + j\theta$ and $\gamma = j$.

The codeword is formed by the combination of four information symbols which are to be decoded together. Due to this, the possible number of codewords is large which makes the exhaustive search through the whole lattice unfeasible and thus define the bottleneck in the practical implementation of the MIMO systems. One of the first algorithms to cater this problem is the sphere decoding algorithm, which is discussed next.

Note that, to facilitate the sphere decoding, all the modules of the modeled system are to be implemented in the real domain [7]. The complex matrices are vectorized by separating real and imaginary parts and stacking them one above the other to form a real array. For the purpose of illustration, the
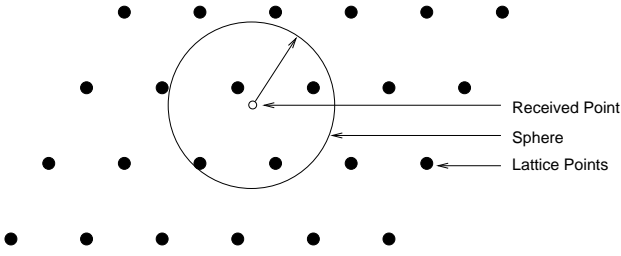
Fig. 1. Geometrical representation of the lattice, sphere and the received point.

equivalent 8×1 real vector $\mathbf{X}_{re}$ of the 2×2 complex codeword $\mathbf{X}$ is shown below:

$$\mathbf{X}_{re} = \frac{1}{\sqrt{5}} \begin{bmatrix} re(\alpha(a+b\theta)) \\ im(\alpha(a+b\theta)) \\ re(\alpha(c+d\theta)) \\ im(\alpha(c+d\theta)) \\ re(\bar{\alpha}(\gamma(c+d\bar{\theta}))) \\ im(\bar{\alpha}(\gamma(c+d\bar{\theta}))) \\ re(\bar{\alpha}(a+b\bar{\theta})) \\ im(\bar{\alpha}(a+b\bar{\theta})) \end{bmatrix}. \tag{2}$$

III. SPHERE DECODING ALGORITHM

In the case of independent fading channels with perfect information of the channel at the receiver, the Maximum Likelihood (ML) decoding requires the minimization of the following metric:

$$m(\mathbf{X}|\mathbf{Y}, \mathbf{H}) = ||\mathbf{Y} - \mathbf{HX}||, \tag{3}$$

where $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{H}$ are the code word, received point and the channel matrix, respectively. The dimension of these matrices is defined by the number of transmit and receive antennas in the modeled MIMO system. This suggests a straight-forward solution of calculating the Euclidean distance from all the possible lattice points and declaring the nearest as the estimate of the transmitted signal. This is true in principle but can be computationally very complex when the total number of lattice points possible is very high (e.g., $4^8 = 65536$ for 16-QAM information symbols in Golden Code).

As discussed above, the exhaustive search through the whole lattice $\{\mathbf{HX}\}$ has an exponential complexity and is not a feasible solution [3]. The first major breakthrough was the introduction of sphere decoding algorithm, which has a cubic computational complexity. This algorithm searches through the lattice to find those points which lie inside the sphere of the pre-defined radius (r) centered at the received point. In a two-dimension problem illustrated in Fig. 1, one can easily restrict the search by drawing a circle around the received signal just enough to enclose a few lattice points and eliminate the search of all the points outside the circle. Only the points lying inside the sphere are taken for further metric calculations to find the estimate of the transmitted point [7]. Alternatively, only those points are considered, which satisfy the following equation:

$$||\mathbf{Y}_{re} - \mathbf{H}_{re}\mathbf{X}_{re}|| \leq r, \tag{4}$$

where $\mathbf{Y}_{re}$ and $\mathbf{H}_{re}$ are the real vector equivalents of $\mathbf{Y}$ and $\mathbf{H}$ respectively. They are calculated by the same procedure as explained for $\mathbf{X}_{re}$ for the Golden Code.

Though this approach has a much lower computational complexity than the exhaustive search, it still needs to be reduced to facilitate the application of the lattice decoders in the practical MIMO systems.

IV. THE PROPOSED QUADRANT-LOCALIZED SEARCH ALGORITHM

The proposed quadrant-localized search algorithm takes into account the directional attributes of the received point to reduce the computational complexity of the existing decoders significantly. This is achieved by reducing the number of lattice points to be considered for the metric calculations by localizing the search in a single quadrant. For this purpose, the direction cosines are used to find the angle of the received vector with all the axis and then deciding the quadrant in which it lies.

In the present work, the proposed scheme is applied to the conventional sphere decoder by using the implementation method introduced in [7]. First, for finding the sphere decoder bounds, each complex codeword is vectorized into a real vector by separating the real and imaginary parts of all the elements of the complex codeword matrix and stacking them one above the other. Consequently, the lattice $\{\mathbf{HX}\}$ is mapped into $\{\mathbf{H}_{re}\mathbf{x}_{re}\}$. Further, the lattice points can be written as the set $\{\mathbf{H}_{re}\mathbf{x}_{re} = \mathbf{uM}\}$, where $\mathbf{M}$ is the lattice generator matrix and $\mathbf{u} = (u_1, u_2 \ldots u_n) \in \mathbb{Z}^n$ is the integer vector to which the information bits are mapped. In the case of golden code, the integer vector $\mathbf{u}$ can be directly related to the information symbols as follows:

$$\mathbf{u} = \begin{bmatrix} re(a) & im(a) & re(b) & im(b) & re(c) & im(c) \\ re(d) & im(d) \end{bmatrix}. \tag{5}$$

Physically, the lattice $\{\mathbf{H}_{re}\mathbf{x}_{re}\}$ (denoted henceforth as Φ) can be thought of as a result of linear transformation defined by the lattice generator matrix $\mathbf{M} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ over a cubic lattice $\{\mathbf{u}\}$. The problem then can be confined to find a lattice point nearest to the received point and can be mathematically expressed as:

$$\min_{\mathbf{v} \in \Phi} ||\mathbf{y}_{re} - \mathbf{v}|| = \min_{\mathbf{w} \in \mathbf{y}_{re} - \Phi} ||\mathbf{w}||. \tag{6}$$

Thus we have to find the shortest vector $\mathbf{w}$ in the transformed lattice $\mathbf{y}_{re} - \Phi \in \mathbb{R}^n$ as given in [7]. $\mathbf{y}_{re}$ can be defined in terms of the lattice generator matrix as $\mathbf{y}_{re} = \rho\mathbf{M}$ with $\rho = (\rho_1, \rho_2 \ldots \rho_n) \in \mathbb{R}^n$ being the received counterpart of $\mathbf{u}$. The real vector $\mathbf{w}$ is similarly transformed by the lattice generator matrix as $\mathbf{w} = \xi\mathbf{M}$ with $\xi = (\xi_1, \xi_2 \ldots \xi_n) \in \mathbb{R}^n$ being a real vector. By using the above definitions of ξ_i, ρ_i and u_i, we can write $\xi_i = \rho_i - u_i$, for $i = 1, 2 \ldots n$. It defines the translated coordinate axis in the space of cubic lattice $\{\mathbf{u}\}$. As a result of this transformation, the sphere of radius r centered at the received point is transformed to an ellipsoid centered at origin in the new coordinate system. The general expression for the range of points lying inside the sphere (and hence the ellipsoid) can be found in [7].

Predefined: Unit vectors $\hat{x}_i$, along all the coordinate axes.

1. Input data: ρ and the error margin Δ.
2. Find the direction cosines ($\cos \theta_i$) of the received points as given by (7).
3. Find err = $1/\cos(\pi/2 - \Delta)$.
4. Calculate: $\cos \theta_i \leftarrow sign(\cos \theta_i)\frac{\lfloor |\cos \theta_i . err|\rfloor}{err}$.
5. IF $\cos \theta_i = 0$; don't localize and search in both the directions of the axis;
6. ELSE localize in one direction of the axis according to the following:
   (a) IF $\cos \theta_i > 0$; search in the positive side of the axis;
   (b) ELSE search in the negative side of the axis ENDIF.
7. ENDIF.
8. Repeat the steps 2 to 7 for all the coordinate axes.
9. END PROCEDURE.

After confining the lattice search inside the sphere by following the above approach, we then limit the search in the quadrant in which the received point lies. The algorithm to be followed for this purpose is summarized in Table 1. Initially, the direction cosines between the transformed lattice point (ρ) and the unit vector along i^{th} axis ($\hat{x}_i$) are defined as follows:

$$\cos \theta_i = \frac{\langle \rho, \hat{x}_i \rangle}{||\rho||}, \qquad (7)$$

for $i = 1, 2, \ldots, n$, where $\langle . \rangle$ denotes inner product and $||.||$ denotes the Euclidean norm.

If $\cos \theta_i$ is positive, we just consider the sphere points lying on the positive side of the ith coordinate axis. If $\cos \theta_i$ is negative, points lying on the negative side are considered. The procedure is repeated for all the n-axes to localize the algorithm in a particular quadrant. This is found to reduce the complexity of the sphere decoder significantly because all the points lying inside the sphere but outside the quadrant in which the received point lies, are not considered for metric evaluation. If the sphere itself lies only in one quadrant, the proposed and the sphere algorithm have the same complexity. However, the probability of the occurrence of such a case is seen to be very low in the practical situations.

One practical case occurs when the received vector is very close to the origin, resulting in the direction cosines to be highly prone to noise. In such cases, even a small noise can change the direction of the vector drastically. This case is taken care by defining a threshold distance d_0 from the origin. If the distance of the received vector is less than this threshold, all the sphere points are considered for matric evaluation and it is not localized in a particular quadrant. The inclusion of this in our algorithm have seen significant improvement in the performance. However, we are still working on some theoretical aspects of deciding the threshold value (d_0) to maximize the performance gain.

Another important practical issue to be taken into account is the sensitivity of the direction cosine values. If the direction cosine in a particular case is close to zero, i.e., θ_i is nearly equal to $\pi/2$, the received point is equally probable to lie on the either side of the coordinate axis. To take this fact into account, we define an error margin Δ and don't localize in

any direction if $\theta_i \in [\pi/2 - \Delta, \pi/2 + \Delta]$. The choice of the error margin Δ is a design issue and is discussed in detail in the next section along with the simulation results. Steps 3 and 4 of the proposed algorithm (given in Table 1), outline the method followed to take into account the error margin in the present work.

A. Graphical Interpretation of the Proposed Algorithm

The implementation of this algorithm can be graphically thought of as tree search where each path through the tree corresponds to a possible transmitted vector. The proposed algorithm acts as a pruning algorithm where a sub-tree can be rejected on any depth depending upon the violation of the boundary conditions imposed by the sphere decoding algorithm and the direction cosines. At this point, the tree is backtraced and a different branch is tested. This is explained in detail with reference to Fig. 2 which presents an example of the path followed by the search algorithm. For the purpose of illustration, we consider the case of decoding only 2 2-QAM symbols at a time.

First level ($l = 1$) corresponds to the bounds on u_n. Only those nodes which satisfies the aforementioned conditions are traversed by the algorithm and the nodes which violate this condition are ignored altogether along with their sub-trees. Traversing the first valid node (node 1 in Fig. 2) at $l = 1$, the algorithm enters the subtree and finds the bounds on u_{n-1} for that node. This is continued until any of the condition is violated (node 3 in Fig. 2). The algorithm then searches for next valid node in the same level and traverses its sub-tree until the condition is again violated (node 9 in Fig. 2) or it reaches the node belonging to the lowermost level (node 21 in Fig. 2). If no valid node is available at that level, the algorithm backtraces to the higher levels in search of the valid node.

When the algorithm is able to traverse any node of the lowermost level, the transmitted vector corresponding to the path of this node from the root node (at $l = 1$) is said to lie inside the sphere. Hence, the number of points considered for matric evaluation is equal to the number of nodes of the lowermost level traversed.

V. RESULTS AND DISCUSSION

The modeled system uses 16-QAM symbols for the golden encoder, which defines the cardinality of the codeword alphabet $\mathbf{X}$ to be 4^8. The performance comparison of the conventional sphere decoder and the proposed Quadrant-Localized Sphere Decoder (QLSD) with various values of Δ is presented in Fig. 3. As theoretically expected, the performance degrades when we don't keep any error margin in the direction cosine values ($\Delta = 0$). The performance nearly approaches that of sphere decoder when sufficient error margin is chosen. The computational complexity comparison between the sphere decoder and the proposed QLSD in terms of the number of points taken for metric evaluation is presented in Fig. 4. The computational complexity is defined in terms of these points because their enumeration involves most of the major computational operations, whereas other points just involve comparison operation which can be simply ignored. With
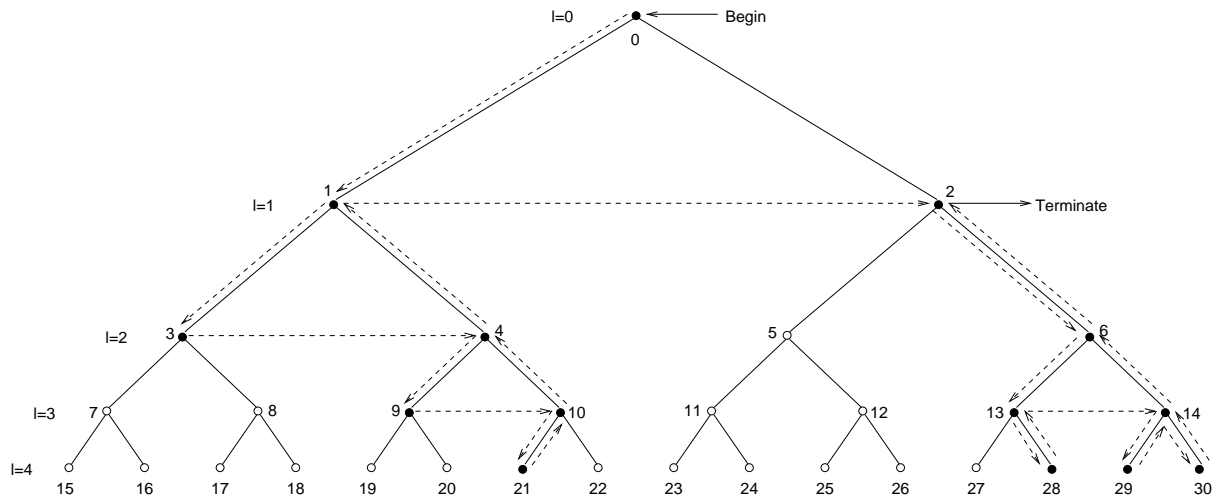
Fig. 2. Graphical interpretation of the proposed algorithm as a tree search algorithm.

the increasing error margin, the number of points chosen for metric calculations increase but are still significantly less than those in the sphere decoder.

VI. CONCLUSION

A quadrant localized search based lattice decoding algorithm for a full-rate space time code, i.e., golden code, is proposed. The algorithm is sufficiently general to be applied to any popular lattice decoding algorithm to reduce its complexity further. We have specifically applied it to the conventional sphere decoder using the golden space time encoder and have observed a significant complexity reduction with only a small degradation in the bit error rate at high SNR. The performance of the proposed algorithm on the sphere decoder using an adaptive radius value is currently being investigated by the authors.

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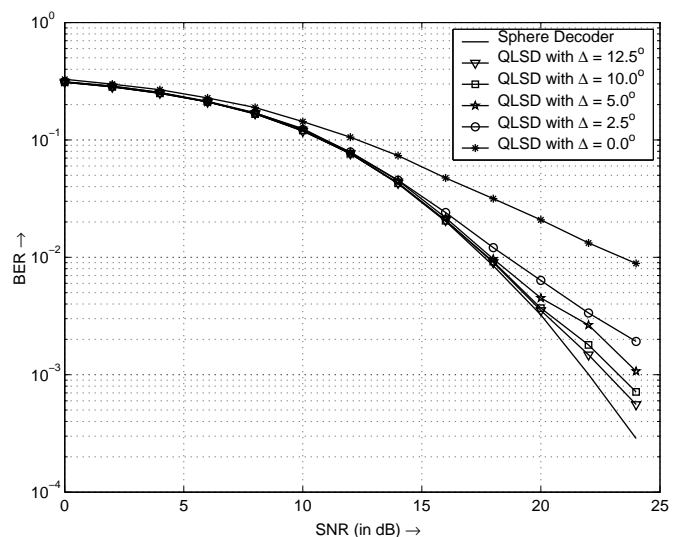
Fig. 3. Performance comparison of the sphere decoder and the proposed QLSD with various Δ values.
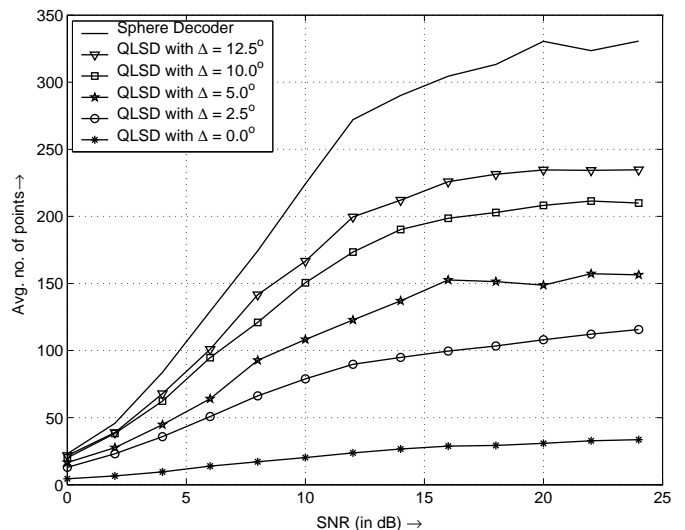

Fig. 4. Computational complexity comparison of the sphere decoder and the proposed QLSD with various Δ values.